A Non-Convex PDE Scale Space

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Abstract. For image filtering applications, it has been observed recently that both diffusion filtering and associated regularization models provide similar filtering properties. The comparison has been performed for regularization functionals with convex penalization functional. In this paper we discuss the relation between non-convex regularization functionals and associated time dependent diffusion filtering techniques (in particular the Mean Curvature Flow equation). Here, the general idea is to approximate an evolution process by a sequence of minimizers of iteratively convexified energy (regularization) functionals.

Keywords: Morphological regularization, diffusion filtering, equivalence relations.

1 Introduction

Let $A : X \to 2^X$ be a *maximal monotone* operator on a real Hilbert space X. Here, we call A maximal monotone, if for every $x, x' \in X$ the implication

$$x' \in Ax \quad \Leftrightarrow \quad \langle x' - Ay, x - y \rangle \geq 0 \text{ for every } y \in X$$

holds. Then there exists a solution of

$$\frac{du}{dt}(t) + A(u(t)) \ni 0 \quad (t \geq 0), \quad u(0) = u^0 . \tag{1}$$

For the precise mathematical formulation of this statement we refer to Zeidler [1, Theorem 32.P]. The solution of (1) is given by

$$u(t) = \lim_{\mathcal{N} \to \infty} \left(I + \frac{t}{\mathcal{N}} A \right)^{-\mathcal{N}} u^0 .$$

See e.g. Crandall & Liggett [2]. We define

$$u_k^{\mathcal{N}} := \left(I + \frac{t}{\mathcal{N}} A \right)^{-k} u^0 \quad (k = 0, 1, \ldots, \mathcal{N}) \text{ and } u^{\mathcal{N}} := u_{\mathcal{N}}^{\mathcal{N}} .$$

From this formula it is evident that $u_k^{\mathcal{N}}$ solves

$$u + \frac{t}{\mathcal{N}} A(u) \ni u_{k-1}^{\mathcal{N}} \quad (k = 1, \ldots, \mathcal{N}) \ . \tag{2}$$

An important example of a maximal monotone operator is the subdifferential $A = \partial J$ of a convex functional $J : X \to \mathbb{R} \cup \{+\infty\}$ defined on a real Hilbert space X. In this case (1) is a *gradient flow equation* and $u_k^{\mathcal{N}}$ minimizes the functional

$$u \to \frac{1}{2} \| u - u_{k-1}^{\mathcal{N}} \|_{L^2(\Omega)}^2 + \frac{t}{\mathcal{N}} J(u) \quad (k = 1, 2, \ldots, \mathcal{N}) \ . \tag{3}$$

That is, the solution of the gradient flow equation can be approximated by iterative regularization.

In [3–5] we performed a systematic comparison of regularization, iterative regularization, and the solution of the according gradient flow equation for image filtering. The experiments show similar solutions for all three methods. Recently Mrázek, Steidl, and Weickert [6, 7] proved analytically for the one-dimensional discrete bounded variation functional $J(u)$ that both regularization and the solution of the discretized gradient flow equation are exactly the same. The similarity relation between the three methods has been validated for gradient flow equations with $A = \partial J$ maximal monotone (which follows from the convexity of J). In this paper we show that the solution of the *Mean Curvature Motion (MCM)*

$$\frac{dv}{dt}(t) = |\nabla v(t)| \nabla \cdot \left(\frac{\nabla v(t)}{|\nabla v(t)|} \right) \quad (t > 0) \ , \quad v(0) = v^0 \ , \tag{4}$$

is approximated by the $\mathcal{N}$-th minimizer of a *non-convex* iterative regularization technique, where in each iteration step a regularization parameter $\alpha = T/\mathcal{N}$ is used. Here, in contrast to (2) we determine $u_k^{\mathcal{N}}$ by solving an equation of the form

$$u + \frac{t}{\mathcal{N}} A_{\frac{t}{\mathcal{N}}}(u) \ni u_{k-1}^{\mathcal{N}} \ .$$

Note that the operator A now depends on $t/\mathcal{N}$. Provided that the limit $u^{\mathcal{N}} = u_{\mathcal{N}}^{\mathcal{N}}$ exists for $t/\mathcal{N} \to 0$, we expect to have a solution of

$$\frac{\partial u}{\partial t}(t) \in - \lim_{s \to 0+} A_s(u(t)) \ .$$

This provides a formal relation between the Mean Curvature Flow equation by mimicking nonlinear semi-group theory.

The MCM equation has been extensively studied. For instance, it is well-known that it attains a unique *viscosity solution* for given continuous and bounded initial data $v^0 : \mathbb{R}^n \to \mathbb{R}$ (see e.g. Evans [8]). Only in very special cases the solution can be calculated analytically. Invariance properties and the use of MCM for image processing applications have been studied by Alvarez & Guichard & Lions & Morel [9]. MCM is an example of a morphological filtering technique.

Therefore, we call the associated non-convex variational principle investigated in this paper *morphological regularization* method.

In [10] a variational form related to the mean curvature flow equation has been derived and a relaxation technique has been used to prove existence of a generalized minimizer. This approach is impractical for a numerical solution since the functional has to be redefined via Γ-limits first, and the relaxed functional eventually has to be minimized. The variational formulation reveals interesting properties (see [11]): it can be motivated as a regularization functional to clean noisy images with random perturbations of the level lines.

The outline of this paper is as follows: In Section 2 we recall the formal relation between the Mean Curvature Flow equation and the according variational principle. In Section 3 we prove well-posedness of iterative regularization based on the concept of convexification. Moreover, a nontrivial part is the characterization of the relaxed functional on the nonreflexive Banach space of functions of bounded variation. Previously, we computed the convex envelope for approximations on Sobolev spaces (see [13, 12, 11]). In Section 4 we discuss the numerical minimization of the nonconvex variational principle and review solving the Mean Curvature Flow equation. The results extend previous numerical experiments in [10] for the minimization of the variational principles, which have been implemented for relatively large regularization parameters. In Section 5 we compare iterative regularization and the solution of the Mean Curvature Flow equation.

2 The Link between MCM and Iterative Regularization

In order to establish the link relation between Mean Curvature Flow and variational forms we study the following energy functional:

$$I(u) := I_{\alpha,u^0}(u) := \int f\big(x, u(x), \nabla u(x)\big)\, dx \quad (\alpha > 0)\ , \tag{5}$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$f(x, \xi, A) = \frac{\big(\xi - u^0(x)\big)^2}{2|A|} + \alpha|A|\ . \tag{6}$$

We can interpret I as a regularization functional with *fit-to-data* term $\int \frac{(u-u^0)^2}{2|\nabla u|}$ and the total variation semi-norm as *fidelity term*.

Aside from the theoretical interest in this functional we use it for solving imaging problems with discontinuous solutions. This motivates the usage of the *total variation semi-norm* for penalization, which has turned out to be quite successful for this purpose (cf. Rudin & Osher & Fatemi [14, 15]).

The following computations are purely formal and not mathematically rigorous. The steepest descent direction of the functional I is

$$\partial I(u) := \frac{u - u^0}{|\nabla u|} + \nabla \cdot \left(\left(\frac{(u - u^0)^2}{2|\nabla u|^2} - \alpha \right) \frac{\nabla u}{|\nabla u|} \right)\ . \tag{7}$$

Therefore, a minimizer of I satisfies the optimality condition

$$u + \alpha A_\alpha(u) := u + \alpha|\nabla u|\nabla \cdot \left(\left(\frac{(u-u^0)^2}{2\alpha|\nabla u|^2} - 1 \right) \frac{\nabla u}{|\nabla u|} \right) \ni u^0 \ . \tag{8}$$

We set $\alpha = t/\mathcal{N}$ and perform iterative regularization by minimization of the functionals $I^k_{t/\mathcal{N}}$, $(k = 1, \ldots, \mathcal{N})$, defined by $I^k_{t/\mathcal{N}} := I_{t/\mathcal{N}, u^{\mathcal{N}}_{k-1}}$ with $u^{\mathcal{N}}_0 := u^0$. The minimizer of $I^k_{t/\mathcal{N}}$ is denoted by $u^{\mathcal{N}}_k$.

With the change of notation $\Delta T := T/\mathcal{N}$, $v(T) := u^{\mathcal{N}}_{\mathcal{N}}$, $v(T - \Delta T) := u^{\mathcal{N}}_{\mathcal{N}-1}$, we find from the according optimality condition for the functional $I^{\mathcal{N}}_{T/\mathcal{N}}$ (cf. (8)) which we multiply by $|\nabla v(T)|/\Delta T$ that

$$\frac{v(T) - v(T - \Delta T)}{\Delta T} \in |\nabla v(T)|\nabla \cdot \left(A(T, \Delta T, v) \frac{\nabla v(T)}{|\nabla v(T)|} \right) \ , \tag{9}$$

where

$$A(T, \Delta T, v) := 1 - \frac{\Delta T}{2} \frac{(v(T) - v(T - \Delta T))^2}{(\Delta T)^2} \frac{1}{|\nabla v(T)|^2} \ .$$

Taking $\Delta T \to 0^+$ and considering $\frac{dv}{dt}(T) = \lim_{\Delta T \to 0^+} \frac{v(T) - v(T - \Delta T)}{\Delta T}$, we recover (4).

For the regularization functional (3), if J is convex, there exists a unique minimizer of the associated regularization functional. Here this is no longer trivial and is a first step of an analysis.

3 Minimizers of Non-Convex Energy Functionals

In this section we prove existence of a minimizer of the functional

$$I(u) := \int_\Omega \frac{\big(u(x) - u^0(x)\big)^2}{2|\nabla u(x)|} \, dx + \alpha |Du|(\Omega) \quad (\alpha > 0) \ . \tag{10}$$

Here Ω is a bounded domain with Lipschitz boundary and $|Du|(\Omega)$ denotes the *total variation* semi-norm. By Du we denote the distributional derivative of u, which is a Radon measure on Ω. Thus we can use the Lebesgue decomposition $Du = \nabla u \, dx + D^s u$, where $\nabla u \in L^1(\Omega)$ denotes the absolutely continuous part of Du and $D^s u$ is the singular part (cf. Rudin [19])[1]. In (10) we define

$$\frac{\big(u(x) - u^0(x)\big)^2}{2|\nabla u(x)|} := 0 \text{ if } u(x) = u^0(x) \, .$$

Minimization of the functional I is considered over the space $\mathrm{BV}(\Omega)$, the space of functions of bounded variation (cf. Evans & Gariepy [16] or Ambrosio & Fusco & Pallara [17]). There are two major difficulties associated with the functional:

[1] We follow the terminology of Ambrosio & Fusco & Pallara [17] and call $D^s u$ the singular part. Other publications denote by $D^s u$ the jump part of the distributional gradient, which belongs to discontinuities in the function u. In particular, in this paper $D^s u$ also contains the Cantor part of u

1. For a convex function g and a measure m the functional $J(m) = \int_\Omega g\big(m(x)\big)$ is well-defined (see e.g. Temam [18]). Here, this theory is not applicable, since the functional I is non-convex with respect to the measure Du, the derivative of the function $u \in BV(\Omega)$.
2. The functional I is *not* lower semi-continuous with respect to the weak* topology on $\mathrm{BV}(\Omega)$, and compensated compactness arguments are not applicable to prove existence of a minimizer.

A standard approach to obtain a meaningful interpretation of I is via *relaxation* (cf. [20]). For a functional $J : X \to \mathbb{R} \cup \{\infty\}$ and $\emptyset \neq X \subseteq \mathrm{BV}(\Omega)$, and $u \in \mathrm{BV}(\Omega)$ its relaxation is defined by

$$\mathcal{R}(J,X)(u) := \begin{cases} +\infty \text{ if } u \notin \overline{X} \cap \mathrm{BV}(\Omega) \\ \inf\big\{\liminf_{k\to\infty} J(u^{(k)}) : \{u^{(k)}\} \subset X, \|u^{(k)} - u\|_{L^1(\Omega)} \to 0\big\} \end{cases} \tag{11}$$

Here $\overline{X}$ is the closure of X with respect to the $L^1(\Omega)$-norm. In order to simplify the notation we define $\mathcal{R}(I) := \mathcal{R}\big(I, \mathrm{BV}(\Omega)\big)$. In the following we show that $\mathcal{R}(I)$ attains a minimizer that can be considered a generalized minimizer of I.

Theorem 1. *Let $u^0 \in L^\infty(\Omega)$, then the functional $\mathcal{R}(I)$ attains a minimizer in $\mathrm{BV}(\Omega)$ that can be considered a generalized minimizer of I, i.e., if the minimum of I is attained in $u \in \mathrm{BV}(\Omega)$, then u is a minimizer of $\mathcal{R}(I)$.*

Proof. The functional $\mathcal{R}(I)$ is lower semi-continuous with respect to the L^1-topology on $\mathrm{BV}(\Omega)$, coercive, and proper (i.e., $\mathcal{R}(I) \not\equiv \infty$). Thus it attains a minimizer in $\mathrm{BV}(\Omega)$. To see that $\mathcal{R}(I)$ is proper take $u(x) = x_1$ if $x = (x_1, \ldots, x_n)$. Then $|\nabla u(x)| = 1$. Thus, $I(u) < \infty$ and consequently $\mathcal{R}(I) < \infty$ showing that $\mathcal{R}(I)$ is proper. The coercivity assertion follows from the characterization of $\mathcal{R}(u)$ given in Theorem 2. To show that each minimizer of I is a minimizer of $\mathcal{R}(I)$ we take $c := \inf\{I(u)\}$. The definition of the relaxed functional implies that $\inf\{\mathcal{R}(I)(u)\} \geq c$. Since I attains the minimum value c, we also have that $\mathcal{R}(I)(u) \leq c$ by using the constant sequence $\{u\}$ in the right hand side of (11).

We now turn to characterizing the relaxed functional.

Theorem 2. *If $u^0 \in L^\infty(\Omega)$, then*

$$\mathcal{R}(I)(u) = I_c(u) := \int_\Omega f_c\big(x, u(x), \nabla u(x)\big)\, dx + \alpha |D^s u|(\Omega) \quad (u \in \mathrm{BV}(\Omega)) \,. \tag{12}$$

Here $Du = \nabla u\, dx + D^s u$ is the Lebesgue decomposition of the distributional gradient of u and

$$f_c(x,\xi,A) := \begin{cases} \dfrac{\big(\xi - u^0(x)\big)^2}{2|A|} + \alpha|A| \,, & \text{if } \sqrt{2\alpha}|A| > |\xi - u^0(x)| \,. \\ \sqrt{2\alpha}|\xi - u^0(x)| \,, & \text{if } \sqrt{2\alpha}|A| \leq |\xi - u^0(x)| \end{cases} \tag{13}$$

Before we prove this theorem, we require some properties of the function f_c, which are summarized in the following lemma:

Lemma 1. *Let $u^0 \in L^\infty(\Omega)$. For almost every $x \in \Omega$*

(a) $f_c(x,\cdot,\cdot)$ *is convex,*
(b) $f_c(x,\cdot,\cdot)$ *is continuously differentiable in every point* $(\xi, A) \neq \big(u^0(x), 0\big)$.

Proof. For $x \in \Omega$ let $U_1 := \{(\xi, A) : \sqrt{2\alpha}|A| < |\xi - u^0(x)|\}$ and $U_2 := \{(\xi, A) : \sqrt{2\alpha}|A| > |\xi - u^0(x)|\}$. For $(\xi, A) \in U_1$ we have

$$\nabla f_c(x, \xi, A) := \nabla_{\xi,A} f_c(x, \xi, A) = \left(\sqrt{2\alpha}\,\mathrm{sgn}\big(\xi - u^0(x)\big), 0\right),$$

and for $(\xi, A) \in U_2$ we have

$$\nabla f_c(x, \xi, A) = \left(\frac{\xi - u^0(x)}{|A|}, \left(\alpha - \frac{\big(\xi - u^0(x)\big)^2}{2|A|^2}\right)\frac{A}{|A|}\right).$$

For $\sqrt{2\alpha}|A| - |\xi - u^0(x)| \to 0$ both gradients coincide, and thus f_c is continuously differentiable. Obviously $f_c(x,\cdot,\cdot)$ is convex on U_1. Since the Hessian of $f(x,\cdot,\cdot)$ is positive definite, $f_c(x,\cdot,\cdot)$ is convex on U_2. From [21, Sec. 42, Thm. B] it follows that the differentiable function f_c is convex, iff ∇f_c is monotone, i.e., $\big(\nabla f_c(x, \xi, A) - \nabla f_c(x, \zeta, B)\big) \cdot \big((\xi, A) - (\zeta, B)\big) \geq 0$ for all (ξ, A), (ζ, B). Since f_c is continuously differentiable and monotone on $\mathrm{int}(U_1)$ and $\mathrm{int}(U_2)$ it follows that ∇f_c is monotone on $\mathrm{int}(\bar{U}_1 \cup \bar{U}_2) = \mathbb{R} \times \mathbb{R}^n$, which shows the convexity of f_c.

From Lemma 1 it follows that the operator $\int_\Omega f_c(x, u(x), v(x))\,dx$ is well-defined for $u, v \in L^1(\Omega) \times \big(L^1(\Omega)\big)^n$. In particular $\int_\Omega f_c(x, u(x), \nabla u(x))\,dx$ is well-defined, if $u \in L^1(\Omega)$ and ∇u is the absolutely continuous part of Du.

Proof (of Theorem 2). Let

$$I^*(u) := \begin{cases} \int_\Omega f\big(x, u(x), \nabla u(x)\big)\,dx & \text{for } u \in W^{1,1}(\Omega), \\ +\infty & \text{else.} \end{cases}$$

It is immediate that $I(u) \leq I^*(u)$, and since $f_c \leq f$ we also have $I_c(u) \leq I(u)$. Consequently, it follows that

$$\mathcal{R}(I_c)(u) \leq \mathcal{R}(I)(u) \leq \mathcal{R}(I^*)(u). \tag{14}$$

Therefore, to prove the assertion of this theorem, it suffices to show that $\mathcal{R}(I^*)(u) = I_c(u)$. Since $I^*(u) = +\infty$ for $u \notin W^{1,1}(\Omega)$, we have

$$\mathcal{R}(I^*)(u) = \mathcal{R}\big(I^*, W^{1,1}(\Omega)\big)(u).$$

Every $u \in \mathrm{BV}(\Omega)$ can be approximated by a sequence $\{u^{(k)}\}_{k\in\mathbb{N}} \subset W^{1,1}(\Omega)$ satisfying $\|u^{(k)} - u\|_{L^1(\Omega)} \to 0$. Moreover, from the definition of $\mathcal{R}(I^*)$ it follows that for every $k \in \mathbb{N}$ there exists $\tilde{u}^{(k)} \in W^{1,1}(\Omega)$ satisfying $\|\tilde{u}^{(k)} - u^{(k)}\|_{L^1(\Omega)} \leq 1/k$ and

$$\mathcal{R}(I^*)(u^{(k)}) \geq I(\tilde{u}^{(k)}) - 1/k. \tag{15}$$

For $u \in W^{1,1}(\Omega)$ it follows from the general results in [22] that

$$\mathcal{R}(I^*)(u) = \mathcal{R}\big(I^*, W^{1,1}(\Omega)\big)(u) = I_c(u) \,. \tag{16}$$

From (15), (16), and the fact that $\|\tilde{u}^{(k)} - u\|_{L^1(\Omega)} \to 0$, it follows that

$$\begin{aligned}\mathcal{R}(I^*)(u) \le \liminf_{k\to\infty} I^*(\tilde{u}^{(k)}) = \liminf_{k\to\infty} I(\tilde{u}^{(k)}) \le \\ \le \liminf_{k\to\infty} \mathcal{R}(I^*)(u^{(k)}) = \liminf_{k\to\infty} I_c(u^{(k)}) \,.\end{aligned}$$

Thus, $\mathcal{R}(I^*)(u) = \mathcal{R}\big(I_c; W^{1,1}(\Omega)\big)(u)$ for $u \in \mathrm{BV}(\Omega)$. We note that for $u \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$ and $\varepsilon > 0$, we may choose a sequence $u^{(k)} \in W^{1,1}(\Omega)$ satisfying $I_c(u^{(k)}) \to \mathcal{R}(I^*)(u)$, which satisfies $\|u^{(k)}\|_{L^\infty} < \|u\|_{L^\infty} + \varepsilon$ for all $k \in \mathbb{N}$. In other words, setting $X^r := \{u \in \mathrm{BV}(\Omega) : \|u\|_{L^\infty} < r\}$ we have

$$\mathcal{R}(I^*)(u) = \mathcal{R}\big(I_c; X^r \cap W^{1,1}(\Omega)\big)(u) \text{ for } u \in X^r \,. \tag{17}$$

For $r > 0$ and $u \in W^{1,\infty}(\Omega)$ let

$$f^r(x,\xi,A) := \begin{cases} \dfrac{\big(\xi - u^0(x)\big)^2 \wedge r^2}{2|A|} + \alpha|A| \,, & \text{if } \sqrt{2\alpha}|A| > |\xi - u^0(x)| \wedge r \,, \\ \sqrt{2\alpha}\big(|\xi - u^0(x)| \wedge r\big) \,, & \text{if } \sqrt{2\alpha}|A| \le |\xi - u^0(x)| \wedge r \,, \end{cases}$$

and

$$I_c^r(u) := \int_\Omega f^r\big(x, u(x), \nabla u(x)\big)\, dx \,.$$

Here $a \wedge b, a \vee b$ denote the minimum, maximum of a and b, respectively. Since $\|u^0\|_{L^\infty} =: r_0 < \infty$ it follows that for every $u \in \mathrm{BV}(\Omega)$ satisfying $\|u\|_{L^\infty} < r - r_0$ we have $I_c(u) = I_c^r(u)$. Thus, from (17) we find that for $u \in X^{r-r_0}$

$$\mathcal{R}(I^*)(u) = \mathcal{R}\big(I_c^r; X^r \cap W^{1,1}(\Omega)\big)(u) \,.$$

Using [23, Thm. 4.1.4] it follows that for $u \in X^{r-r_0}$ we have

$$\mathcal{R}(I^*)(u) = \int_\Omega f^r\big(x, u(x), \nabla u(x)\big)\, dx + |D^s u|(\Omega) = I_c(u).$$

Using [24, Prop. 2.4] it follows that for every $u \in \mathrm{BV}(\Omega)$

$$\mathcal{R}(I^*)(u) = \lim_{r\to+\infty} \mathcal{R}(I^*)\big((u \wedge r) \vee -r\big) = \lim_{r\to\infty} I_c\big((u \wedge r) \vee -r\big) \,.$$

From this and the monotone convergence theorem (see e.g. [16]) the assertion follows. □

We recall that the assumption $u^0 \in L^\infty(\Omega)$ is needed in order to satisfy the growth conditions required in [23]. The functional $\mathcal{R}(I)$ is coercive with respect to the total variation semi-norm. It can be shown by a truncation argument that there exists a minimizer of $\mathcal{R}(I)$ with L^∞-norm less than $\|u^0\|_{L^\infty(\Omega)}$. Thus the functional attains a minimizer u in BV.

We note that convexification of non-convex functionals on BV is a recent research topic. We mention the papers [23, 25–27].

4 Numerics

We describe a finite element method for minimization of the functional $\mathcal{R}(I)(u)$. We use $M := (n-1) \times (m-1)$ quadratic finite elements $(Q_i)_{i=1,\ldots,M}$ to cover Ω and bilinear basis functions $(\phi_j)_{j=1,\ldots N:=n\times m}$ which are centered at the corner points of the finite elements. We denote by $\mathcal{Q} := \operatorname{span}\{Q_i : i = 1, \ldots, M\}$. The initial data u^0 is given as discrete values on a rectangular grid of size $n \times m$ and is identified with the function $u^0 = \sum_{i=1}^N u_i^0 \phi_i$.

The minimizer u_{NCBV} (non-convex bounded variation) of the functional $\mathcal{R}(I)$ solves the optimality condition $\partial\mathcal{R}(I)(u_{\mathrm{NCBV}}) = 0$, where $\partial\mathcal{R}(I)$ is the subgradient of $\mathcal{R}(I)$. In the weak form the optimality condition reads as

$$\begin{aligned} \frac{u-u^0}{|\nabla u|}\phi_j + \left(\alpha - \frac{(u-u^0)^2}{2|\nabla u|^2}\right)\frac{\nabla u \nabla \phi_j}{|\nabla u|} &= 0 \quad \text{if } \sqrt{2\alpha}|\nabla u| > |u-u^0| \,, \\ \sqrt{2\alpha}\frac{u-u^0}{|u-u^0|}\phi_j &= 0 \quad \text{if } \sqrt{2\alpha}|\nabla u| \le |u-u^0| \,, \end{aligned} \tag{18}$$

where $j = 1, \ldots, N$. The second equation implies that if $\sqrt{2\alpha}|\nabla u| \le |u-u^0|$, then $u(x) = u^0(x)$, from which it follows that $|\nabla u(x)| = 0$. With the abbreviation

$$a(u) = \frac{1}{|\nabla u|} \wedge \frac{\sqrt{2\alpha}}{|u-u^0|} \,, \quad b(u) = \left(\left(\alpha - \frac{|u-u^0|^2}{2|\nabla u|^2}\right) \vee 0\right)\frac{1}{|\nabla u|}$$

equation (18) reads as follows

$$\sum_{i=1}^N \int_\Omega a(u)\phi_i\phi_j\, u_i + b(u)\nabla\phi_i\nabla\phi_j u_i = \int_\Omega a(u)\phi_i\phi_j u_i^0 \quad (j = 1, \ldots, N) \,. \tag{19}$$

Let $U := (u_1, \ldots, u_N)^T$, $U^0 := (u_1^0, \ldots, u_N^0)^T$,

$$M_{ij} := \int_\Omega \phi_i\phi_j \text{ and } L_{ij} := \int_\Omega \nabla\phi_i\nabla\phi_j \,.$$

We approximate $a(u)$ and $b(u)$ by elementwise constant functions $\tilde{a}(U)$ and $\tilde{b}(U)$.

Using this notation and these approximations, (18) reads as

$$\left(\tilde{a}(U_{\mathrm{NCBV}})M + \tilde{b}(U_{\mathrm{NCBV}})L\right) U_{\mathrm{NCBV}} = \tilde{a}(U_{\mathrm{NCBV}})M\, U^0 \,. \tag{20}$$

This system is solved applying the fixed point iteration:

$$\begin{aligned} &\tilde{a}(U_{\mathrm{NCBV}}^{(s)})\, M\, U_{\mathrm{NCBV}}^{(s+1)} + \tilde{b}(U_{\mathrm{NCBV}}^{(s)})\, L\, U_{\mathrm{NCBV}}^{(s+1)} \\ &\qquad = \tilde{a}(U_{\mathrm{NCBV}}^{(s)})\, M\, U_{\mathrm{NCBV}}^0 \quad (s = 0, 1, 2, \ldots) \,. \end{aligned} \tag{21}$$

The iteration is terminated if a given tolerance tol is reached, i.e., if $|U_{\mathrm{NCBV}}^{(s+1)} - U_{\mathrm{NCBV}}^{(s)}| \le \mathrm{tol}$ or s exceeds a given limit. In each iteration step, for solving the linear system for $U^{(s+1)}_{\text{NCBV}}$ we use the C(onjugate)G(radient)-method. In order to avoid occurring oscillations, the following modified scheme can be used: For $s = 0, 1, 2, \ldots$, solve

$$\tilde{a}(U^{(s)}_{\text{NCBV}})\, M\, U^{(*)}_{\text{NCBV}} + \tilde{b}(U^{(s)}_{\text{NCBV}})\, L\, U^{(*)}_{\text{NCBV}} = M\, U^{(s)}_{\text{NCBV}} \tag{22}$$

due to the unknown function $U^{(*)}_{\text{NCBV}}$ and using the solution set

$$U^{(s+1)}_{\text{NCBV}} = U^{(s)}_{\text{NCBV}} + \delta^s\, (U^{*}_{\text{NCBV}} - U^{(s)}_{\text{NCBV}}) \quad (s = 0, 1, 2, \ldots)\,, \tag{23}$$

where $0 < \delta^s \leq 1$ tends to zero for increasing s.

5 Results

In this section we show that the iterated solution of $\mathcal{R}(I)$ gives similar results as solving the MCM equation. We show that $v(T)$, the solution of the MCM equation and $u^{\mathcal{N}}_{\mathcal{N}}$ are almost identical. We recall that $u^{\mathcal{N}}_k$ is the minimizer of the functional $\mathcal{R}(I)$ where u^0 is replaced by $u^{\mathcal{N}}_{k-1}$, $k = 1, \ldots, \mathcal{N}$ and $\alpha = T/\mathcal{N}$.

The MCM equation at time $T = \Delta T \tilde{\mathcal{N}}$ is calculated by solving the system of equations (note that ΔT needs not be identical to α)

$$\tilde{c}(U_{\text{MCM}})(M + \Delta T\, L)U_{\text{MCM}} = \tilde{c}(U_{\text{MCM}})MU^{k-1}_{\text{MCM}} \quad (k = 1, \ldots, \tilde{\mathcal{N}}) \tag{24}$$

and denoting the solution by U^k_{MCM}. A vector U_{MCM} is associated with the function $u_{\text{MCM}} = \sum_{i=1}^{N} (U_{\text{MCM}})_i \phi_i$ from which an approximation $\tilde{c}(U_{\text{MCM}})$ for $c(u) = \frac{1}{|\nabla u|}$ is determined that is piecewise constant on the finite elements. I.e., $\tilde{c}(U_{\text{MCM}})|_{Q_{ij}} = c(u_{\text{MCM}})(p_{ij})$, where p_{ij} is the midpoint of cell Q_{ij}. The implemented FE-Method for solving the Mean Curvature Motion essentially follows [28].

For fixed k, we again use a fixed point iteration to solve (24):

$$\tilde{c}(U^{(s)}_{\text{MCM}})(M + \Delta T\, L)U^{(s+1)}_{\text{MCM}} = \tilde{c}(U^{(s)}_{\text{MCM}})MU^{k}_{\text{MCM}} \quad (s = 0, 1, \ldots)\,. \tag{25}$$

If $\|U^{(s+1)}_{\text{MCM}} - U^{(s)}_{\text{MCM}}\| <$ tol the iteration is terminated and $U^{k+1}_{\text{MCM}} := U^{(s+1)}_{\text{MCM}}$.

In the following we present two numerical comparisons of regularization, i.e., minimizing the functional (10), iterative regularization, and solving the MCM equation (4).

In the first numerical experiment we have calculated the solution of the MCM equation at time $T = 20$. We use a step length $\Delta T = 0.25$. The iterative regularization has been implemented with $\alpha = T/\mathcal{N}$ and varying parameters $\mathcal{N} = 2, 10, 20, 40$. The comparison shows the original image, MCM filtered image and the iterated regularized solution $u^{\mathcal{N}}$ for various parameters $\mathcal{N}$. As $\mathcal{N}$ increases, iterative regularization approximates the solution of the MCM equation. The second example is a comparison between MCM and regularization, i.e., we compare the solution of the MCM equation at time $T = 10, 50, 100, 300$ with u^1, i.e., the minimizer of (5) with $\alpha = 10, 50, 100, 300$.

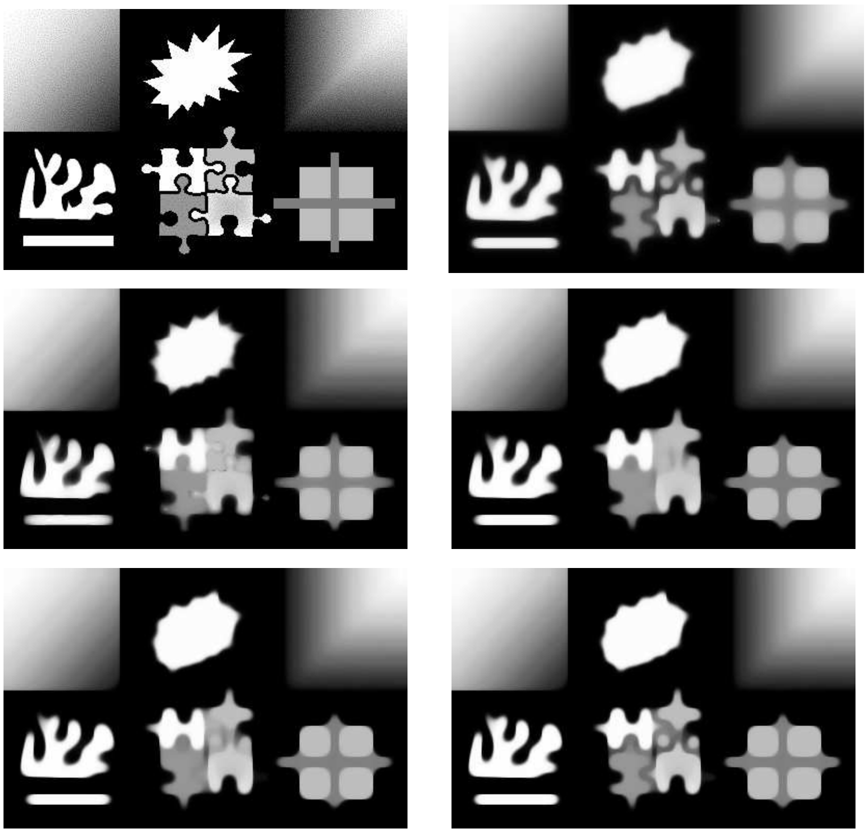

Fig. 1. *Top Left*: Original data, *Top Right*: Solution of the Mean Curvature equation at time $T = 20$. Iterative Regularization. Images show $u_{\mathcal{N}}^{\mathcal{N}}$. $\mathcal{N} = 2(\alpha = 10)$ *(Middle Left Column)*, $\mathcal{N} = 10(\alpha = 2)$ *(Middle Right Column)*, $\mathcal{N} = 20(\alpha = 1)$ *(Bottom Left Column)*, $\mathcal{N} = 40(\alpha = 0.5)$ *(Bottom Right Column)*.

6 Conclusion

In this paper we have generalized the concept of gradient flow equations with subdifferentials of convex functionals to non-convex functionals. The general idea is to approximate an evolution process by a sequence of minimizers of iteratively convexified energy (regularization) functionals. Although there is no mathematical theory for "non-convex" gradient flow equations, the results in this paper show the similar filtering behavior. The results of this paper have been formulated exemplarily for the Mean Curvature equation but can be generalized to other well known equations in morphological image analysis, such as the *affine invariant Mean Curvature* equation (cf. [10]). For gradient flow equations with subdifferential of a convex functional it has been observed recently that both

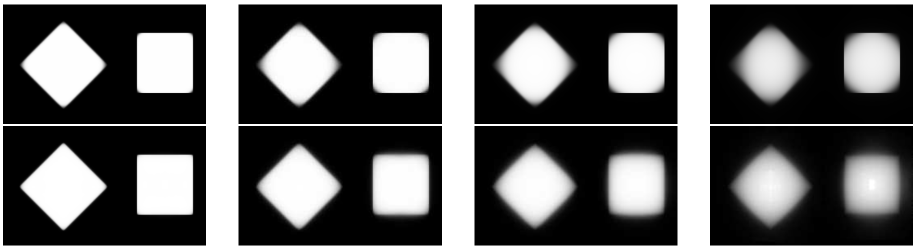

Fig. 2. *Top:* solution of MCM equation at $T = 10, 50, 100, 300$ and *Bottom*: u^1, the minimizer of (5), with $\alpha = 10, 50, 100, 300$.

diffusion filtering and associated regularization models provide similar filtering properties. Here this analogy has been shown for the Mean Curvature Flow equation and the associated *non-convex* energy formulation.

Acknowledgement

This work has been supported by FWF (Austrian Science Fonds) Projects Y-123INF and P15617. The work of F.L. is supported by the Tiroler Zukunftsstiftung. A.O. is supported by the Tiroler Zukunftsstiftung, by the Austrian Federal Ministry for Economic Affairs and Labour and by the Upper Austrian Government within the framework "Industrial Competence Centers".

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